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Generalized Sectional Convergence and Multipliers

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DEDICATED TO PROFESSOR CHARLES W. MCARTHUR AND PROFESSOR ALBERT (TOMMY) WILANSKY, TWO MENTORS WHO HAD A PROFOUND INFLUENCE ON BOTH OF US.

We define a generalized sectional convergence scheme (gscs) as a sequence (T_n) of finitely non-zero matrices which converges coordinatewise to the identity matrix. If $T_n x$ converges to x for each x in a topological sequence space S, then we say that S has $AK\ (T_n)$, a generalization of sectional convergence (AK). We prove that a generalization of the Dieudonné Weak Basis Theorem is valid in this new context. A gscs (T_n) together with a dense subspace Λ of l^1 determines a matrix space $M = \Lambda(T_n)$. If Λ is barrelled and each member of M is a matrix representation of a linear operator T which maps the locally convex K space S into itself (denote this by $MS \subset S$), then we can draw conclusions about topological and approximation properties of S. For an appropriate type of (T_n) we will have the following: If S is an FK space with AD and $MS \subset S$, then S has the generalized sectional convergence associated with (T_n). If S is a sequence space, which has AD in the $\beta\varphi$ topology and $MS \subset S$, then S is barrelled in the $\beta\varphi$ topology. The many dense barrelled subspaces of l^1 are examples of dense $\beta\varphi$ subspaces Λ of l^1, and the latter still support our conclusions even though Λ and the corresponding multiplier space M may be very small. This reduction of M is novel even in the context of ordinary sectional convergence.

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1. Introduction

The study of the relations between the multipliers of a sequence space S and its topological properties has a long history in functional analysis. Although this topic certainly has older roots, we first cite the 1934 paper [15] of Köthe and Toeplitz. In this work the authors characterize a "*normal*" sequence space S as one for which $(u(n)s(n)) \in S$ for all $s \in S$ and $u \in l^\infty$, and establish some topological properties of normal sequence spaces. Since then, coordinatewise multipliers have appeared in the context of sequence spaces; see, e.g., the paper of Garling [10]; and in the context of Schauder bases, see, e.g., the paper of McGivney and Ruckle [17]. Köthe and Toeplitz also considered problems about the approximation of a sequence by its sections. Zeller [35] formally defined "Abschnitz Konvergenz" (AK) for a sequence space. A topological sequence space S has AK precisely when the coordinate vectors (e_n) form a Schauder basis for S. There is a much greater volume of literature on Schauder bases than on sequence spaces, but the two areas illumine one another [16, 32].

Typical coordinate multiplier theorems are that (e_n) forms a basis of an *AD FK* space S if and only if $bvS \subset S$ where bv is the space of sequences of bounded variation, and that (e_n) forms an unconditional basis of S if and only if $l^\infty S \subset S$. There are also statements of these theorems in the context of Schauder bases. See the works of Kadec [13], Yamazaki [34], and McGivney and Ruckle [17]. Kadec and Pełczynski [14] determined that under certain conditions (using the language of biorthogonal sequences and Schauder bases) (e_n) will form an unconditional basis for an *FK* space S if and only if $m_0 S \subset S$ where m_0 is the subspace of l^∞ consisting of finitely valued sequences. In obtaining their results they often assumed the biorthogonal sequence under examination was norming. We shall define a sequence space analog of the norming property in Section 2 after explaining our terminology. Kadec and Pełczynski conjectured that their result would hold for all *FK* spaces. Bachelis and Rosenthal [1] verified this. See also the papers of Bennett and Kalton [3–5]. The key ingredient of the solution obtained by Bachelis and Rosenthal is a consequence of a theorem of Seever [31] which implies that m_0 is a barrelled subspace of l^∞. This provides an example of improving a multiplier inclusion theorem by reducing the required multiplier space.

Buntinas [6] and Meyers [18] both envisioned and studied more general convergence conditions than AK for *FK* spaces. Gaposhkin and Kadec [12] made similar innovations in the language of Schauder bases. In Section 3 of this paper we shall define a generalized sectional convergence scheme, a concept which resembles the operational basis of Gaposhkin and Kadec, and show how we can represent the dual of spaces which have such a generalized sectional convergence. The work of this section generalizes that found in [20] where we studied the duals of spaces with various kinds of bases. One of our results (Theorem 3.8(d)) generalizes Dieudonné's version of the Weak Basis Theorem [8] to the setting of generalized convergence and also extends it to a class larger than the barrelled AD spaces.

In [11] Garling proved a theorem which stated in part that $bv_0S = S$ if and only if S has AK and is barrelled in a certain topology. Since the hypothesis requires equality instead of inclusion it is very strong; the conclusion is also strong. This theorem raises the possibility of a different type of multiplier result, namely, $MS \subset S$ imples S is barrelled in some topology or other. The topology which we use is a natural sequence space topology, which we call the *beta-phi* ($\beta\varphi$) *topology*. We introduced this topology in [23] and studied it in other papers. In Section 2 we shall define the ($\beta\varphi$) topology on a sequence space and prove a couple of basic facts.

In the fourth section we define a type of matrix multiplier space $\Lambda(T_n)$ that arises from a generalized sectional convergence scheme and a subspace Λ of l^1. We are particularly interested in the case when Λ is a dense barrelled subspace Λ of l^1 with respect to the Banach space topology because in this case $\Lambda(T_n)$ will also be barrelled so that we can apply variants of the Closed Graph Theorem and Uniform Boundedness Principle. Fortunately, Bennett, e.g. [2], and others, e.g. [24, 26] have found many examples of such Λ.

The main results of our paper appear in Section 5 that characterizes generalized sectional convergence in terms of multipliers. Section 5 also contains multiplier criteria sufficient to assure that a sequence space is barrelled in its $\beta\varphi$ topology. We generalize existing multiplier results of the form "$bv_0S \subset S$" in two ways.

1. We use hypotheses of form $MS \subset \tilde{S}$ where M can be a space of matrices. If the matrices in this space M are diagonal this hypothesis reduces to one about coordinatewise multipliers.

2. We can reduce the size of M via small subspaces of l^1. Even for the case of reducing bv_0 this seems to be an innovation.

We also generalize the sufficiency of Garling's factorization theorem. For R a barrelled sequence space with generalized sectional convergence and S any sequence space, we conclude that the space generated by RS is barrelled and has the same generalized sectional convergence as R in the $\beta\varphi$ topology.

2. Sequence Spaces and the Beta-Phi Topology

This section will introduce our notation and the definition of the beta-phi ($\beta\varphi$) topology on a sequence space. Our main result is that every row finite matrix between two sequence spaces having their $\beta\varphi$ topologies is continuous. Although this result is merely a special case of a more general result about spaces in duality, it will be helpful later.

By a *sequence space* we mean a linear space S of real or complex sequences. We write a scalar sequence s as a function on the positive integers

$$s = (s(n)) = (s(1), s(2), \ldots),$$

so we can reserve subscripts for sequences of such sequences. We denote by e_n the sequence for which $e_n(n) = 1$ and $e_n(m) = 0$ for $m \neq n$. A K *space* is a locally convex space of sequences S such that each coordinate functional E_i defined by $E_i(s) = s(i)$ is continuous. In this paper we shall also assume that a K space contains each e_n. A few common K spaces, to which we shall refer, are:

ω the space of all sequences;

φ the space of all sequences which are eventually 0, i.e., $s \in \varphi$ if there is an integer $M(s)$ such that $s(i) = 0$ for $i > M(s)$;

l^1 the space of all sequences s such that $\Sigma_n |s(n)| < \infty$;

c_0 the space of all sequences s which converge to 0;

bv the space of all sequences s such that $\Sigma_n |s(n+1) - s(n)| < \infty$;

bv_0 the space of all sequences in bv which converge to 0;

l^∞ the space of all bounded sequences.

We shall write infinite matrices as functions on the product of positive integers. Thus $A = (A(i, j))$ is the matrix with entry $A(i, j)$ in the jth column of the ith row. If the linear span φ of the set $\{e_n\}$ is dense in a K space S, then we say S has AD. If $\{e_n\}$ forms a Schauder basis of S, that is, if $s = \Sigma_n s(n)e_n$ for each $s \in S$ with the convergence in the topology of S, then we say S has AK.

For a sequence s we define $P_n s$ to be the sequence which coincides with s in the first n coordinates and is 0 thereafter. Thus we have

$$P_n s(i) = \begin{cases} s(i) & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases} \quad \text{and} \quad P_n s = \sum_{j=1}^{n} s(j)e_j.$$

The infinite diagonal matrix with 1's in the first n positions and 0's elsewhere represents the operator P_n as a mapping from ω into φ. We denote this matrix by P_n as well.

There is a natural duality between every sequence space S and the space φ given by the equation

$$\langle s, t\rangle = \sum_j s(j)t(j), \qquad s \in S, t \in \varphi.$$

Since $t \in \varphi$, the series contains only finitely many non-zero terms. A subset B of φ is called *S-bounded* if

$$p_B(s) = \sup\{|\langle s, t\rangle| : t \in B\} < \infty$$

for each $s \in S$; i.e., if the set

$$B^\circ = \{r \in \omega : |\langle r, t\rangle| \leq 1 \text{ for each } t \in B\}$$

absorbs points of S. A K space S is said to have a φ topology if the topology on S is determined by a family of seminorms $\{p_B\}$ as B ranges over a subcollection of S-bounded subsets of φ. This is equivalent to there being in S a base of 0-nbds which are closed in the relative topology of ω on S.

If B is any subset of φ define S_{p_B} to be the set of all $s \in \omega$ such that $p_B(s) < \infty$. It is not hard to show that S_{p_B} is an FK space with topology determined by coordinatewise convergence plus the seminorm p_B. If S is any K space with a φ topology then S is a topological subspace of the space

$$S_\Xi = \bigcap \{S_{p_B} : B \in \Xi\},$$

where Ξ denotes a *basic* collection of S-bounded subsets of φ by which we mean Ξ has the property that if A is any S-bounded subset of φ there is some $B \in \Xi$ that absorbs A. The closure T of S in the complete space S_Ξ is a completion of S which is a K space (Proposition 3.4 of [21].) If S is a K space there can be at most one complete K space T that contains S as a dense subspace and induces the original topology on S, since any such T is continuously included in the Hausdorff space ω. If such a space T exists then we call T the *K space completion* of S and write $\hat{S} = T$. Section 3 of the paper [21] discusses basic properties of K spaces having a K space completion, there called "consistent" spaces.

The $\beta\varphi$ topology is the locally convex topology on S determined by the collection of all S-bounded subsets of φ. If S has its $\beta\varphi$ topology we let $\tilde{S}$ denote its K space completion. Most familiar sequence spaces bear the $\beta\varphi$ topology, e.g., φ with the strongest locally convex topology, the l^p spaces $1 \leq p \leq \infty$, and ω with the topology of coordinatewise convergence; but not l^p $(0 < p < 1)$ with the non-locally convex complete metric linear topology. The concept of φ topology is related to the concept of norming biorthogonal sequence; see, e.g., [23]. Every absorbing absolutely convex subset of a K space S, which is closed in the relative topology of ω, is a barrel. It follows that if S is barrelled, its topology must be at least as strong as the $\beta\varphi$ topology.

The barrelled AD space S presented in Example 3.6 contains a Cauchy net which does not converge in S but does converge in ω to some $u \in S$. It follows that S is a barrelled K space having no K space completion. Moreover, the topology on S is strictly finer than the $\beta\varphi$ topology since the $\beta\varphi$ topology is consistent.

If S is a K space then by S^f we denote the space of all sequences $(f(e_n))$ as f ranges over the space S' of all continuous linear functionals on S. The sequence space S^f is a representation for the dual space of φ endowed with the relative topology of S or the dual of the closure of φ in S with this relative topology. See [23]. If S has a φ topology then S^f is the union of the coordinatewise closures of all subsets B of φ such that the seminorm p_B is continuous in the topology of S [23].

We shall use the following convention when describing spaces: When we use a modifier in front of the word space, we endow the space automatically with all of the attributes used to define that modifier. For example, if we say "Let S be a $\beta\varphi$ space" this will mean the same as "Let S be a K space with the $\beta\varphi$ topology."

THEOREM 2.1. *If a row finite matrix A maps the sequence space S into the sequence space T then A maps S continuously into T with respect to the $\beta\varphi$-topologies on S and T.*

Proof. Since every row of A belongs to φ, it follows that $A^{\top}$, the transpose of A, maps φ into φ. If $B \subset \varphi$ is T-bounded then $A^{\top} B$ must be S-bounded since we have

$$\begin{aligned}\sup\{|\langle u, s\rangle| : u \in A^{\top}(B)\} &= \sup\{|\langle A^{\top} v, s\rangle| : v \in B\}\\ &= \sup\{|\langle v, As\rangle| : v \in B\} < \infty\end{aligned}$$

for each $s \in S$. If $C = A^{\top}(B)$, the preceding equalities show that

$$p_B(As) = p_C(s)$$

so that A maps S continuously into T. ∎

Let S be a subspace of a K space T. We say S is a *barrelled* (resp. $\beta\varphi$) *subspace of T* if, in the relative topology, S is barrelled (resp. has its $\beta\varphi$ topology).

PROPOSITION 2.2. (a) *If S is a dense barrelled subspace of a $\beta\varphi$ space*

T, then S is a $\beta\varphi$ subspace (and T is barrelled). (b) *If S is a dense $\beta\varphi$ subspace of a K space T, then T is a $\beta\varphi$ space and every S-bounded subset of φ is T-bounded.*

Proof. Let A be an arbitrary S-bounded subset of φ. As A° is closed in ω and S and T are continuously included in ω, $U = A^\circ \cap S$ and $W = A^\circ \cap T$ are absolutely convex and closed in S and T, respectively, and W contains the closure V of U in T. Now U is a barrel in S, and under the hypothesis of either (a) or (b) is a 0-neighborhood in S. By density, the closure V is a 0-neighborhood in T and thus so is W. As W is absorbing in T, A is T-bounded. Thus S and T determine the same bounded subsets of φ. This is part of the conclusion of (b), and from it follows the main conclusion of (a). (Any locally convex space containing a dense barrelled subspace is, itself, barrelled.)

To finish the proof of (b), note that since U is a basic 0-neighborhood in S, so is V in T, and $V \subset W$ implies the K space topology on T is finer than its $\beta\varphi$ topology. Now either topology on T makes S a dense $\beta\varphi$ subspace, and the infimum of the two topologies is Hausdorff, so that they both yield the same dual T', and thus the same closed absolutely convex sets. Hence V is a basic neighborhood of 0 in both topologies on T, and they coincide. ∎

PROPOSITION 2.3. *Let (x_n) be a sequence of elements in a locally convex space X. The following statements are equivalent:*

1. *(x_n) is bounded in X;*
2. *the partial sums of the series $\sum s(n)x_n$ form a Cauchy sequence in X for all $s \in l^1$;*
3. *the partial sums of the series $\sum s(n)x_n$ form a Cauchy sequence in X for all s in some dense barrelled subspace Λ of l^1;*
4. *the partial sums of the series $\sum s(n)x_n$ form a Cauchy sequence in X for all s in some dense $\beta\varphi$ subspace Λ of l^1;*
5. *$(s(n)x_n)$ is bounded in X for all s in some dense $\beta\varphi$ subspace Λ of l^1.*

Proof. Since each of the series is absolutely Cauchy, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$ follows via Proposition 2.2(a).

$5 \Rightarrow 1$. Since weakly bounded implies bounded in a locally convex space it will suffice to prove that $(f(x_n))$ is bounded for all $f \in X'$. Given $s \in \Lambda$, the sequence $(f(s(n)x_n)) = (\langle s, f(x_n)e_n\rangle)$ is bounded because f is continuous and $(s(n)x_n)$ is bounded by hypothesis. This shows that $(f(x_n)e_n)$ is bounded at points of l^1 by Proposition 2.2(b), making it norm-bounded in l^∞ via the Uniform Boundedness Principle. Therefore, $(f(x_n))$ is bounded. ∎

3. Generalized Sectional Convergence

In this section we define generalized sectional convergence for a K space, and show that a barrelled K space having such a convergence must have the $\beta\varphi$ topology. We discuss the dual space of a K space S with generalized sectional convergence and relate it to natural duals of S arising from the convergence scheme. Theorem 3.8 generalizes the Weak Basis Theorem of Dieudonné [8] in two directions, (1) by weakening the requirement that the space be barrelled and (2) by obtaining the conclusion for a more general class of convergence schemes than sectional convergence.

A *generalized sectional convergence scheme* (gscs) means a sequence (T_n) of infinite matrices which has the following properties:

(A) Each T_n has only finitely many non-zero entries;

(B) for each i, j,

$$\lim_n T(i,j) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

It is clear from the definition that if (T_n) is a gscs then so is the sequence (T_n^{T}) of transposes. If S is a K space then each T_n maps S continuously into itself because each of the finitely many non-zero rows of T_n is a finite linear combination of the coordinate functionals. We say that a K space S has $AK(T_n)$ if $\lim_n T_n x = x$ in the topology of S for each $x \in S$. If each T_n is upper triangular we will call (T_n) an *upper triangular* gscs, if each T_n is lower triangular we will call (T_n) a *lower triangular* gscs; if each T is diagonal we will call (T_n) a *diagonal sectional convergence scheme* (dscs). If $T_n = P_n$ for each n then $AK(T_n)$ coincides with the usual condition AK. Finally, we say that a sequence space S has AD if the space φ is dense in S.

A traditional series to sequence summability method will lead to a diagonal sectional convergence scheme (dscs). If A denotes the matrix for the method, then the dscs consists of the diagonal matrices T_n for which $T_n(i, i) = A(n, i)$. Thus the diagonal of T_n is the nth row of A. This comes about because if you apply A to the formal sum $\sum_j s(j)e_j$, you will obtain the sequence $\sum_j A(n, j)s(j)e_j = (A(n, j)s(j))$ in the *nth* row.

As an example we construct a family of diagonal sectional convergence schemes (dscs) in the following way. For each real number $t \geq 0$ let $P_n^{(t)}$ denote the diagonal matrix for which

$$P_n^{(t)}(i, i) = \left(\frac{n - i + 1}{n}\right)^t \qquad \text{if } i \leq n \text{ and } 0 \text{ otherwise}.$$

Then we see $(P_n^{(0)})$ gives the usual sectional projections, (P_n). For any sequence s the sequence (of sequences) $(P_n^{(1)}s : n = 1, 2, \ldots)$ gives the first averages of $(P_n s : n = 1, 2 \ldots)$, and $(P_n^{(k)}s : n = 1, 2 \ldots)$ the averages of $(P_n^{(k-1)}s : n = 1, 2 \ldots)$.

Both set-wise and topology-wise, φ and ω are extreme opposite examples of K spaces. We easily see that, due to its paucity of elements, φ has $AK(T_n)$ if (T_n) is any upper triangular gscs; and because of its weak topology, ω has $AK(T_n)$ if (T_n) is any lower triangular gscs. Indeed, φ has $AK(T_n)$ if and only if each $I_j = \{i : T_n(i, j) \neq 0 \text{ for some } n\}$ is finite. For, if $T_n e_j \to e_j$ in φ, then $(T_n e_j)_n$ is bounded in φ, and since φ has the strongest locally convex topology, $(T_n e_j)_n$ must be finite-dimensional; i.e., I_j is finite. Conversely, if I_j is finite then $T_n e_j \to e_j$ in S for any K space S by the definition of gscs, since all Hausdorff linear topologies coincide on the finite-dimensional span of $(T_n e_j)_n$. If each I_j is finite (T_n) is said to be a *quasi-upper triangular gscs* (qut). We also note that ω has $AK(T_n)$ if and only if each $J_i = \{j : T_n(i, j) \neq 0 \text{ for some } n\}$ is finite. For, if J_i is finite and $x \in \omega$, then $E_i(T_n x) \to E_i(x)$ by the definition of gscs, and if J_i is infinite, there exists an inductively defined $x \in \omega$ such that $E_i(T_n x) = n$ for infinitely many n. If each J_i is finite, we call (T_n) a *quasi-lower triangular gscs* (qlt). A gscs which is both a quasi-upper triangular gscs (qut) and a quasi-lower triangular gscs (qlt) is said to be a *quasi-diagonal sectional convergence scheme* (qd).

THEOREM 3.1. *φ has $AK(T_n)$ if and only if (T_n) is a quasi-upper triangular gscs (qut), and ω has $AK(T_n)$ if and only if (T_n) is a quasi-lower triangular gscs (qlt).*

THEOREM 3.2. *Suppose S is a $\beta\varphi$ or barrelled space. A gscs (T_n) is equicontinuous on S if and only if $(T_n x)$ is bounded for each $x \in S$.*

Proof. One direction is immediate, since equicontinuous families carry bounded sets onto bounded sets, as is the other for barrelled spaces.

Suppose $(T_n x)$ is bounded for arbitrary x in S, a $\beta\varphi$ space (i.e., with its $\beta(\cdot, \varphi)$ topology). Let A be an S-bounded subset of φ. If $u \in A$ and $n \in \{1, 2, \ldots\}$, then $|\langle x, T_n^{\mathsf{T}} u\rangle| = |\langle T_n x, u\rangle| \leq \sup_k p_A(T_k x) < \infty$. Thus $C = \{T_n^{\mathsf{T}} u : n = 1, 2, \ldots, u \in A\}$ is an S-bounded subset of φ. Each T_n carries $C^{\circ} \cap S$ into $A^{c} \cap S$ and (T_n) is, thus, equicontinuous on S. ∎

We use the standard proof of the Weak Basis Theorem in our next

THEOREM 3.3. *Let S be a K space and (T_n) a gscs.* (a) *S has $AK(T_n)$ if the following three conditions hold*: (i) *(T_n) is equicontinuous*; (ii) *S has AD*; (iii) *$T_n e_j \to e_j$ in S for each j.* (b) *Conversely, if S is a $\beta\varphi$ or barrelled $AK(T_n)$ space, then the three conditions of* (a) *hold.*

Proof. (a) Let U be a neighborhood of 0 and let $x \in S$ be given. Choose a balanced neighborhood V of 0 such that $V + V + V \subset U$. By equicontinuity there is a neighborhood W of 0 such that $T_n(W) \subset V$ for all n. The second condition yields $t \in \varphi$ with $t - x \in W \cap V$. By the third, there is $N > 0$ such that $t - T_n t \in V$ for all $n > N$. Thus $n > N$ implies $x - T_n x = (x - t) + (t - T_n t) + T_n(t - x) \in U$, and so $(T_n x)$ converges to x in S.

(b) Equicontinuity follows from the preceding theorem. ∎

THEOREM 3.4. *If S is an $AK(T_n)$ space and the sequence (T_n) is equicontinuous then the topology on S is a φ-topology.*

Proof. For each closed 0-nbd U of S let $U_* = \cap_n T_n^{-1}(U)$. Since $\{T_n\}$ is an equicontinuous sequence, U_* is a 0-nbd. If x is in U_*, $T_n x$ is in U for each n so x is in U. This means $U_* \subset U$. The collection of all U_* thus forms a neighborhood base at 0 for the topology on S.

It remains to show that each U_* is coordinatewise closed. Suppose u_n is in U_* for each n and u_n converges coordinatewise to $u \in S$. For each k, $(T_k u_n)_n$ is a sequence in U which converges coordinatewise to $T_k u$. Since the range of T_k has finite dimension it follows that $(T_k u_n)_n$ converges to $T_k u$ in S as well. This implies that $T_k u$ is in U for each k so that u is in U_*. ∎

COROLLARY 3.5. *If S is a barrelled $AK(T_n)$ space then S has the $\beta\varphi$ topology.*

While φ and ω have $AK(T_n)$ for every quasi-diagonal sectional convergence scheme (T_n), some barrelled AD spaces S can never have $AK(T_n)$ no matter the choice of the gscs (T_n) as we show in the following.

EXAMPLE 3.6. Let $R \neq \varphi$ be a barrelled $AK(T_n)$ space for some gscs (T_n). Let $u \in R \backslash \varphi$ and choose a linear functional f on R such that $(f(T_n u))$ is unbounded. By a result of Dieudonné (cf. [28]), $N = f^{\perp}$ is barrelled and thus so is the topological direct sum $S = N \oplus L$, where L is any (1-dimensional) algebraic complement to N in R. The topology on S is strictly finer than that on R, which is the $\beta\varphi$ topology. Thus S is not a $\beta\varphi$ space and by Corollary 3.5 fails to be an $AK(U_n)$ space for each gscs (U_n). However, S is still an AD space. For $S' = sp(f) + R'$, and if, for some scalar a and some $g \in R'$, $af + g$ vanishes on φ, then since $(T_n u)$ is bounded in R, $(g(T_n(u)))$ is bounded, so that $((af)(T_n u))$ is also, forcing $a = 0$. Thus $g = 0$ since R is AD, and $af + g = 0$. It follows that φ is dense in S; i.e., S is also an AD space.

The space of the previous example did not have a sequence space completion, but we can also give an example of a BK space that does not have $AK(T_n)$ for any gscs (T_n) even though it has AD and the $\beta\varphi$ topology.

EXAMPLE 3.7. Let E be a separable Banach space which does not have the approximation property [9]. By Theorem 2.1 of [22] E has a complete norming biorthogonal sequence (x_n, f_n). This means that the span (x_n) is dense in E, each $f_n(u) = 0$ only when $u = 0$, $f_m(x_n) = \delta_{mn}$, and there is a subset B of span (f_n) such that

$$p_B(u) = \sup\{|g(u)| : g \in B\}$$

is a norm which determines the topology of S. Let S be the set of all sequences $(f_n(u))$ as u ranges over E. It is clear that the correspondence $u \to (f_n(u))$ is an algebraic isomorphism from E onto S, and that S is an AD BK space if we give it the norm for which this correspondence is an isometry. That S has its $\beta\varphi$ topology then follows since (x_n, f_n) is norming. However, if S were an $AK(T_n)$ space for some gscs (T_n) it would follow that S has the metric approximation property because the sequence (T_n) is an equicontinuous sequence of finite-dimensional operators which converges pointwise to the identity.

For (T_n) a gscs we define the $\beta(T_n)$ *dual* of a sequence space S as the space of all sequences t such that $\lim_n \langle T_n s, t\rangle$ exists for all $s \in S$. After identifying the sequence (T_n) we shall write the $\beta(T_n)$ dual of S as $S^{(\beta)}$. The $\gamma(T_n)$ *dual* $S^{(\gamma)}$ of S consists of all sequences t such that $\sup_n |\langle T_n s, t\rangle| < \infty$ for all $s \in S$. Obviously $S^{(\gamma)}$ contains $S^{(\beta)}$ for every sequence space S. These notations are adaptations of the notations S^β and S^γ which have been used for the β and γ duals relative to the sequence of matrices (P_n) [11].

Dieudonné [8] showed that in a barrelled space a weak basis is a basis if and only if the sequence of partial sums for the expansion of each member of the space is bounded. Part (d) of the following theorem generalizes this result in two directions, (1) by replacing the requirement of being barrelled with a weaker requirement that includes the $\beta\varphi$ topology, (2) by using a quasi-upper triangular gscs instead of the particular gscs (P_n).

THEOREM 3.8. *Suppose S is a K space and (T_n) is a gscs.* (a) *If S is an $AK(T_n)$ space then $S^f \subset S^{(\beta)} \subset S^{(\gamma)}$.* (b) *If S is an $AK(T_n)$ space with the $\beta(S, \varphi)$-topology (a $\beta\varphi$ space) then S is quasi-barrelled.* (c) *If S is a $\beta\varphi$ $AK(T_n)$ space and (T_n) is a quasi-upper triangular gscs then $S^f = S^{(\beta)} = S^{(\gamma)}$.* (d) *If (T_n) is a quasi-upper triangular gscs and S is a barrelled or $\beta\varphi$ AD space and $S^f \subset S^{(\gamma)}$ then S is an $AK(T_n)$ space.*

Proof. (a) If $t \in S^f$ there is a continuous linear functional f_t on S such that $f_t(e_n) = t(n)$ for each n. For each $s \in S$, $T_n s \in \varphi$ for each n so $\lim_n \langle T_n s, t\rangle = \lim_n (f_t(T_n s)) = f_t(s)$ because $\lim_n T_n s = s$ in S.

(b) Let U be a barrel in S which absorbs bounded sets. The set $U_* = \bigcap_n T_n^{-1}(U)$ is thus a barrel since $(T_n s)$ is bounded for all $s \in S$. The set U_* is also contained in U and is coordinatewise closed in S because U is closed. (See the proof of Theorem 3.4.) Therefore, U_* and, hence, U is a 0-nbd in the $\beta\varphi$ topology.

(c) If $t \in S^{(\gamma)}$, then $(T_n^{\mathrm{T}} t)$ is an S-bounded sequence in φ because $\langle s, T_n^{\mathrm{T}} t\rangle = \langle T_n s, t\rangle$. This means that the seminorm p defined by $p(s) = \sup_n |\langle s, T_n^{\mathrm{T}} t\rangle|$ is continuous on S with respect to the $\beta\varphi$ topology. If $s \in \varphi$ then $(T_n s)$ converges to s in φ by Theorem 3.1 so that

$$|\langle s, t\rangle| = |\lim_n \langle T_n s, t\rangle| \le \sup_n |\langle s, T_n^{\mathrm{T}} t\rangle| = p(s).$$

Thus the linear functional defined on φ by $f_t(s) = \langle s, t\rangle$ is continuous on φ with respect to the relative topology induced by S. Since φ is dense in S the functional can be extended continuously to all of S. The extension, which we also denote by f_t, has the property that $f_t(e_n) = t(n)$ for each n. This implies that t must be in S^f.

(d) By Theorem 3.1, each sequence $(T_n e_j)_n$ of the jth columns of (T_n) converges to e_j in φ and thus also in the coarser topology induced on φ by S, so the second and third conditions of Theorem 3.3(a) are satisfied. To demonstrate the first, we need only have $(T_n x)$ bounded for each $x \in S$ because of Theorem 3.2. If $g \in S'$ then $t = (g(e_n)) \in S^f \subset S^{(\gamma)}$ implies $(g(T_n x)) = (\langle T_n x, t\rangle)$ is bounded, and by local convexity the weakly bounded $(T_n x)$ is bounded. ∎

Since every sequentially complete quasi-barrelled space is barrelled, Corollary 3.5 and Theorem 3.8(b) yield

COROLLARY 3.9. *A sequentially complete $AK(T_n)$ space is barrelled if and only if it is $\beta\varphi$.*

While this paper was in press an example [27] was found of a non-barrelled $\beta\varphi$ AK space, so that in Corollary 3.9 and Theorem 3.8(b) "sequentially complete" cannot be omitted and "quasi-barrelled" cannot be replaced by "barrelled."

We now give two examples showing that (d) and (c) of Theorem 3.8 are false if "quasi-upper triangular gscs" is omitted.

Let $1 < p, q < \infty$ with $1/p + 1/q = 1$ and set $S = l^p$ and each $T_n = P_n + W_n$ where W_n is 1 at $(n, 1)$ and 0 elsewhere. Each $T_n : S \to S$ has operator norm $\le \sqrt[p]{2^p + 1}$ so that (T_n) is an equicontinuous gscs, S is a barrelled $\beta\varphi$ AD space, and a routine check shows that $S^f = S^{(\beta)} = S^{(\gamma)} = l^q$. And yet S fails to have $AK(T_n)$, because $\|T_n e_1 - e_1\|_p = \|e_n\|_p = 1$ for all n. Thus (d) of Theorem 3.7 is not valid for arbitrary gscs (T_n).

EXAMPLE 3.10. Let S be the BK space c_0. For each n, let Q_n be a finitely non-zero matrix with the following properties:

1. All entries are ≥ 0, and all columns after the first n are identically 0.
2. For $1 \leq k \leq n$, the sum of the elements of the kth column is $1 - 2^{-k}$.
3. The sum of the elements of each row of Q_n is $\leq n^{-1}$.

This is easily possible and one may even have each Q_n lower triangular, if desired. By 3 the operator norm of each Q_n is $\leq n^{-1}$ on c_0. Thus $(T_n) = (P_n - Q_n)$ is a gscs such that for each $x \in c_0$, $\|T_n x - x\| \leq \|P_n x - x\| + \|Q_n x\| \leq \|P_n x - x\| + n^{-1}\|x\|$, and the limit is 0; i.e., $S = c_0$ has $AK(T_n)$. Let $t = (1, 1, 1, \ldots)$. By 1 and 2, each $T_n^\top t = P_n t - Q_n^\top t = (2^{-1}, \ldots, 2^{-n}, 0, \ldots)$, so that $(\langle T_n x, t\rangle) = (\langle x, T_n^\top t\rangle)$ is convergent for each $x \in S$, and $t \in S^{(\beta)} \subset S^{(\gamma)}$. But it is clear that $t \notin S^f = l^1$. Hence (c) is false if "quasi-upper triangular gscs" is omitted.

EXAMPLE 3.11. If (T_n) is a quasi-upper triangular gscs and S is a $\beta\varphi$ $AK(T_n)$ space then the quasi-barrelled S must have its Mackey topology $\tau(S, S^{(\beta)})$. However, (P_n) is a qut and for $S = l^\infty$ we have $S^\beta = l^1$, and l^∞ with the $\tau(l^\infty, l^1)$ topology has AK, but the $\beta\varphi$ topology on l^∞ is its usual BK topology which does not have AK (or even AD). In short, a Mackey AK space may be non-$\beta\varphi$. Also, metrizable AK topologies on φ provide examples of non-$\beta\varphi$ quasi-barrelled AK spaces. Thus in Corollary 3.5 we cannot replace "barrelled" by either of the more general terms "Mackey" or "quasi-barrelled."

4. $\Lambda(T_n)$-SPACES

In this section we define a type of matrix multiplier space $\Lambda(T_n)$ which arises naturally from a gscs (T_n) and a subspace Λ of l^1. We are particularly interested in such a multiplier space when we can obtain it from a dense subspace Λ of l^1 which is barrelled in the relative topology of l^1. There are many examples of such spaces. Here are five classes:

1. l^p with $0 < p \leq 1$, and the intersection of these spaces [3].
2. Scarce copies of l^1 [2]. Let $u(n)$ be an increasing positive sequence which diverges to ∞. The space l^1_u of all sequences s in l^1 such that

$$\lim_n (\text{number of } \{j : j < n, s(j) \neq 0\})/u(n) = 0$$

is a scarce copy of l^1.

3. Spaces determined by a modulus [25]. A modulus is a non-negative, subadditive function q on $[0, \infty)$ which is continuous and 0 at 0. The space of all sequences s such that $\sum q(s(j)) < \infty$ is barrelled in the relative topology of l^1, but it has a (usually non-locally convex) topology with the metric derived from the modulus.

4. Symmetric subspaces of l^1 which properly contain φ [24].

5. Dilations [26]. If $b \in l^1$ is infinitely non-zero, the dilation space E_b is the span of φ and the vectors $\sum_i b(i)e_{n_i}$ as (n_i) ranges through subsequences of positive integers.

We can augment the ranks of such spaces by adding combinations of the above properties such as scarce copies of spaces determined by a modulus or countable codimensional subspaces of such subspaces that contain φ [28].

All of the spaces Λ described above determine the same bounded subsets of φ as does l^1, namely, the uniformly bounded subsets. Since each space Λ has AK in its $\beta\varphi$ topology it follows from Theorem 3.8(c) that $\Lambda^\gamma = l^\infty$. In [24] it was proved that all *symmetric* $\beta\varphi$ *AD* spaces are barrelled so that every dense symmetric $\beta\varphi$ subspace of l^1 is barrelled. This led us to pose the following problems.

4.1. Problem. If Λ is a subspace of l^1 which determines the same bounded subsets of φ as does l^1, is Λ barrelled in the relative topology of l^1?

4.2. Problem. If Λ is a subspace of l^1 for which $\Lambda^\gamma = l^\infty$, is Λ barrelled in the relative topology of l^1?

While this article was in press, a class of non-barrelled dense $\beta\varphi$ subspaces of l^1 was found [27], negatively answering both problems. Moreover, these subspaces retain the following barrelled-like property useful in Section 5.

(**) (Theorem 2.6 *of* [27]) *If* Λ *is a* $\beta\varphi$ *subspace of* l^1 *and* $B \subset c$ (= *convergent sequences*) *is* Λ*-bounded, then* B *represents an equicontinuous subset of* Λ'; *in particular, if* Λ *is dense in* l^1, *then* B *is bounded in* sup norm.

The hypothesis that Λ be a dense $\beta\varphi$ subspace of l^1 thus becomes viable and more elegant in the context of Section 5. Those who wish this paper to be self-contained may always require that Λ be barrelled, as the authors did originally.

For (T_n) a gscs and Λ a subspace of l^1 we denote by $\Lambda(T_n)$ the space of all infinite matrices of the form

$$\sum_n s(n)T_n, \qquad s \in \Lambda.$$

For each i, j the sequence $(T_n(i, j))$ is bounded so that $\Sigma_n\, s(n)T_n(i, j)$ determines a continuous linear functional on Λ with the relative topology of l^1. Therefore, the space $N(T_n)$ of all s such that $\Sigma_n\, s(n)T_n = 0$ forms a closed subspace of Λ. This implies that with the norm

$$\|T\| = \inf\left\{\sum_n |s(n)| : T = \sum_n s(n)T_n\right\} \tag{1}$$

the space $\Lambda(T_n)$ is a quotient space of Λ. For each i, j we have

$$|T(i,j)| \leq \sup_n\{|T_n(i,j)|\} \sum_n |s(n)|$$

whenever $T = \Sigma_n\, s(n)T_n$. From this we conclude that

$$|T(i, j)| \leq \sup_n\{|T_n(i, j)|\}\|T\|$$

which implies that each coordinate functional $E_{i,j}(T) = T(i,j)$ is continuous on $\Lambda(T_n)$. If Λ is a barrelled subspace of l^1 then $\Lambda(T_n)$ is also barrelled by Corollary 1 on p. 61 of [30].

For two spaces Λ_1 and Λ_2 we may have $\Lambda_1(T_n) = \Lambda_2(T_n)$ as sets but with different topologies determined by the norm (1). Let $t(n) = 2^{-n}$ and let T_n be the diagonal matrix having down the diagonal 1's in the first $n - 1$ places, -1 in the nth place for $n = 1, 2, \ldots$. Set $\Lambda_1 = \varphi$ and $\Lambda_2 = \varphi + \mathrm{sp}(t)$. Since $\Sigma\, t(n)T_n = 0$ we have $\Lambda_1(T_n) = \Lambda_2(T_n)$ as sets. But in $\Lambda_1(T_n)$ we have

$$\left\|\sum_{n=1}^{j} 2^{-n}T_n\right\| = \left\|\sum_{n=1}^{j} 2^{-n}e_n\right\|_1 = 1 - 2^{j}$$

while in $\Lambda_2(T_n)$ we have

$$\left\|\sum_{n=1}^{j} 2^{-n}T_n\right\| = \left\|\sum_{n=1}^{j} 2^{-n}e_n - t\right\|_1 = \sum_{n>j} 2^{-n} = 2^{-j}.$$

Thus the topology on $\Lambda_1(T_n)$ is strictly finer than that on $\Lambda_2(T_n)$.

Also we may have two dense subspaces $\Lambda_1 \neq \Lambda_2$ of l^1, one barrelled and one not, such that $\Lambda_1(T_n) = \Lambda_2(T_n)$ both as sets and topological spaces. Let $T_{2n-1} = T_{2n} = P_n$ for each n. The subspace N of l^1 defined by

$$N = \left\{s \in l^1 : \sum s(n)T_n = 0\right\} = \{s \in l^1 : s(2n - 1) = -s(2n) \text{ for all } n\}$$

is closed in l^1, and

$$M = \{s \in l^1 : s(2n - 1) = 0 \text{ for all } n\}$$

is a closed algebraic complement to N in l^1, and hence is also a topological complement. Let N_0 be a dense $\aleph_0$-dimensional subspace of N and set $\Lambda_1 = M + N_0$ and $\Lambda_2 = M + N$. The subspace Λ_1 is not barrelled since it is metrizable and not quasi-Baire [29], containing the closed $\aleph_0$-codimensional subspace M; and $\Lambda_2 = l^1$ is barrelled. We clearly have

$$\Lambda_1(T_n) \approx \Lambda_1/N_0 \approx M \approx \Lambda_2/N \approx \Lambda_2(T_n).$$

Thus $\Lambda_1(T_n)$ and $\Lambda_2(T_n)$ are the same in every respect and isomorphic to the BK space $l^1 \approx M$, although Λ_1 is non-barrelled and Λ_2 is a Banach space.

PROPOSITION 4.1. *Let (T_n) be a gscs and let Θ be a subspace of l^1. Let Λ denote the largest subspace of l^1 such that, setwise, $\Lambda(T_n) = \Theta(T_n)$. The normed space $\Lambda(T_n)$ is barrelled if and only if Λ is a barrelled subspace of l^1.*

Proof. The space $\Lambda(T_n)$ is isomorphic to the quotient of Λ by the subspace

$$N = \left\{s \in l^1 : \sum s(n) T_n = 0\right\} = \bigcap_{i,j} \{s \in l^1 : \langle s, (T_n(i,j))_n \rangle = 0\},$$

which is a closed subspace of l^1. Thus N is a Banach space and is barrelled. If Λ/N is barrelled, so is Λ since barrelledness solves the three space problem [19]. Conversely, any quotient of a barrelled space is barrelled. ∎

If (T_n) is a dscs then each matrix in $\Lambda(T_n)$ must be diagonal. In this case we can identify $\Lambda(T_n)$ with the space of sequences $\{(U(i, i)) : U \in \Lambda(T_n)\}$ and think of matrix multiplication on a sequence space S by members of $\Lambda(T_n)$ as coordinatewise multiplication of sequences in S and $\Lambda(T_n)$. We adopt this view in considering the family $(P_n^{(t)})$ of diagonal sectional convergence schemes (dscs) introduced in the previous section.

If we make the diagonals of $(P_n^{(t)})$ the columns of a matrix Π_t, then for Λ a subspace of l^1, $\Lambda(P_n^{(t)})$ is equal to $\Pi_t(\Lambda)$, the image of Λ under Π_t. For each t we shall have

$$\Pi_t(i,j) = \begin{cases} \left(\dfrac{j - i + 1}{j}\right)^t & \text{if } i \le j \\ 0 & \text{if } i > j. \end{cases}$$

Considered as a space of diagonals, $l^1(P_n)$ coincides with the space bv_0 of sequences s such that $\lim_n s(n) = 0$ and $\Sigma_n |s(n) - s(n-1)| < \infty$. The space $l^1(P_n^{(1)})$ is the space of all sequences s which converge to 0 and such that $\Sigma_n n|s(n) - 2s(n+1) + s(n+2)| < \infty$. However, the description of the other spaces $l^1(P_n^{(i)})$ does not involve higher differences. The space $l^1(P_n^{(2)})$ consists of all sequences s which converge to 0 and for which

$$\sum_n |s(n) - 4s(n+1) + 7s(n+2) - 8s(n+3) \\ - 7s(n+4) + 4s(n+5) - s(n+6)| < \infty.$$

This was obtained by inverting the matrix Π_2, but details are omitted.

5. Multiplier Theorems

If M is a subset of ω and S is a sequence space then MS denotes the set $\{(u(i)s(i)) : u \in M,\ s \in S\}$. If M is a set of infinite matrices such that the sequence

$$Us = \left(\sum_j U(i,j)s(j)\right)_i$$

is defined for each $U \in M$ and each s in a sequence space S, let MS denote the set $\{Us : U \in M, s \in S\}$. The second notation is a generalization of the first since coordinatewise multiplication of the sequence s by the sequence u is the same as multiplication of s by the diagonal matrix U with $U(i,i) = u(i)$ for all i. Having defined MS let $[MS]$ denote the linear span of $\varphi \cup MS$.

The first result concerns coordinatewise multiplication and generalizes part of Theorem 11 in [11] that requires $bv_0 S = S$ and concludes among other things that S must have AK in the Mackey topology on S determined by S^β.

THEOREM 5.1. *Suppose* (T_n) *is a diagonal sectional convergence scheme and* R *is a* $\beta\varphi$ $AK(T_n)$ *space. For each sequence space* S, *the* $\beta\varphi$ *space* $[RS]$ *has* $AK(T_n)$, *and is barrelled if* R *is.*

Note. For the converse, take $S = \varphi + sp((1, 1, \ldots))$ so that $RS = R$.

Proof. Given $s \in S$, the diagonal mapping $r \to rs$ is continuous from R into $[RS]$ by Theorem 2.1, and thus $\lim(T_n r)\, s = rs$. Since each T_n is diagonal $T_n(rs) = (T_n r)s$ so that $\lim T_n(rs) = rs$ for each $s \in S$ and $r \in R$. By the continuity of addition we conclude $(T_n t)$ converges to t for all $t \in [RS]$.

Suppose R is barrelled and G is an $[RS]$-bounded subset of $[RS]'$. Let $s \in \varphi \cup S$ be given. For $g \in G$ we will also use g to denote the sequence $(g(e_n))$. Composing continuous functions, we define $h_{s,g}$ in R' by $h_{s,g}(r) = g(rs)$, and note that the set $\{h_{s,g} : g \in G\}$ is R-bounded, hence equicontinuous, hence uniformly bounded on the bounded subset $\{T_n r : n = 1, 2, \dots\}$ for each $r \in R$. For each n and each $g \in G$,

$$h_{s,g}(T_n r) = g((T_n r)s) = g(T_n(rs)) = \langle rs, T_n g\rangle$$

so that the subset $C = \{T_n g : g \in G, n = 1, 2, \dots\}$ of φ is bounded at points of $R\varphi \cup RS = \varphi \cup RS$, and is thus $[RS]$-bounded. Therefore, $U = C^\circ \cap [RS]$ is a neighborhood of 0 in $[RS]$, and for $x \in U$ and $g \in G$ we have

$$|g(x)| = \lim|g(T_n x)| = \lim|\langle x, T_n g\rangle| \leq 1;$$

i.e., G is equicontinuous and $[RS]$ is barrelled by Theorem 4, p. 137 of [33]. ▮

The following example shows that our result is a proper extension of that of Garling.

EXAMPLE 5.2. Let S be the space of all sequences t such that $t(\pi(1)) + t(\pi(2)) + \cdots$ converges where π is a fixed permutation of the positive integers for which there is a sequence $w \in S$ such that $\sum w(n)$ diverges to ∞. Let θ denote the inverse permutation of π, and let Q denote the matrix whose nth row is the $\theta(n)$th row of the identity matrix. Then $S = Q(cs)$ where cs denotes the BK AK space of convergent series. Give S the Banach space topology which makes Q an isometry, so that $\|t\| = \sup_n|t(\pi(1)) + t(\pi(2)) + \cdots t(\pi(n))|$. By Corollary 3.5, cs has its $\beta\varphi$ topology, and since Q^{T} maps φ into φ so does S. For each n let u_n denote the sequence for which $u_n(\pi(i)) = 1$ if $i \leq n$ and $u_n(j) = 0$ otherwise. Let (T_n) be the diagonal sectional convergence scheme (dscs) with each T_n having diagonal u_n. Then S has $AK(T_n)$ but not AK, since $\|P_n w\| \geq |\sum_{j=1}^{n} w(j)|$ and the latter quantity grows unbounded. Indeed, one may apply Theorem 5.1: bv_0 is a BK AK space, and the norm on $R = Q(bv_0)$ that makes $Q : bv_0 \to R$ an isometry ensures R is a barrelled $AK(T_n)$ space. Since $bv_0 cs = cs$ it follows that $RS = S = [RS]$. We also note that the space $l^1(T_n)$ may be naturally identified with $R = Q(bv_0)$. Finally, the β dual of S is strictly smaller than the $\beta(T_n)$ dual of S. The Banach Steinhaus Theorem shows that $S^\beta \subset S^f$, and $S^f = S^{(\beta)}$ by Theorem 3.8(c). The contrapositive of part (d) yields $S^f \not\subset S^\gamma$, so that $S^{(\beta)} = S^f \not\subset S^\beta \subset S^\gamma$.

LEMMA 5.3. *Fix (T_n) a gscs, Λ a subspace of l^1, $x \in \omega$, and $i \in \mathcal{N}$.*

(a) *If Λ is a barrelled subspace of l^1 and*

$$f_{x,i}(T_s) = h_{x,i}(s) = \sum_j T_s(i,j)x(j) = E_i(T_s x)$$

converges whenever $s \in \Lambda$ and $T_s = \sum s(n)T_n$, then $f_{x,i}$ and $h_{x,i}$ determine continuous linear functionals on $\Lambda(T_n)$ and Λ, respectively.

(b) *If (T_n) is a quasi-lower triangular gscs (qlt) then the above series always converges and $f_{x,i}$ and $h_{x,i}$ determine continuous linear functionals on $\Lambda(T_n)$ and Λ, respectively.*

(c) *In fact,* (a) *holds if Λ is merely a $\beta\varphi$ subspace of l^1.*

Proof. (a) For $k = 1, 2, \ldots, \sum_{j=1}^k x(j)E_{i,j}$ is continuous on the normed barrelled space $\Lambda(T_n)$, and hence so is the pointwise limit $f_{x,i}$. Moreover, $\|s\| \geq \|T_s\|$ yields $\|h_{x,i}\| \leq \|f_{x,i}\|$.

(b) For arbitrary Λ, the unit ball B of $\Lambda(T_n)$ is contained in the unit ball C of $l^1(T_n)$. If (T_n) is a quasi-lower triangular gscs (qlt), then $f_{x,i}$ is defined on C and is bounded there by (a). Hence $f_{x,i}$ is bounded on B; i.e., continuous on $\Lambda(T_n)$. As above, continuity of $h_{x,i}$ on Λ follows.

(c) Define each sequence t_k so that for all n we have $t_k(n) = \sum_{j=1}^k x(j)T_n(i,j)$. By definition of gscs, $\lim_n t_k(n) = \sum_{j=1}^k x(j)\delta_{i,j}$, and each $t_k \in c$. For $s \in \Lambda$, we have

$$\langle s, t_k\rangle = \sum_{n=1}^{\infty} s(n) \sum_{j=1}^{k} x(j)T_n(i,j) = \sum_{j=1}^{k} x(j) \sum_{n=1}^{\infty} s(n)T_n(i,j).$$

Thus $\lim_k\langle s, t_k\rangle = f_{x,i}(T_s)$ so that $(\langle\cdot, t_k\rangle)_k$ is Λ-bounded, and $M = \sup_k\|\langle\cdot, t_k\rangle\|_{\Lambda'} < \infty$ by (**) in Section 4. Choose $r \in \Lambda$ with $T_r = T_s$ and $\|r\|_1 \leq 2\|T_s\|$. Then

$$|f_{x,i}(T_s)| = |f_{x,i}(T_r)| \leq \sup_k |\langle r, t_k\rangle| \leq M\|r\|_1 \leq 2M\|T_s\|;$$

i.e., $\|f_{x,i}\| \leq 2M$. Again, $\|h_{x,i}\| \leq \|f_{x,i}\|$. ∎

Suppose Λ, (T_n), and S are given and $\Lambda(T_n)S$ is well defined. For $x \in S$ and $v \in \varphi$, define the linear functional $h_{x,v}$ at each $s \in \Lambda$ by

$$h_{x,v}(s) = \left\langle\left(\sum s(n)T_n\right) x, v\right\rangle.$$

Now v is a finite linear combination of certain e_i, and $h_{x,v}$ is the corresponding combination of the $h_{x,i}$ in Lemma 5.3, and so is continuous if either Λ is $\beta\varphi$ or (T_n) is a qlt. If (T_n) is a quasi-lower triangular gscs (qlt) and $\Lambda \supset \varphi$, then $\lim_k h_{x,v}(e_k) = \lim_k\langle T_k x, v\rangle = \langle x, v\rangle$ shows we may view $h_{x,v}$ as a member of c.

THEOREM 5.4. *Suppose (T_n) is a quasi-diagonal sectional convergence scheme (qd) and Λ is a dense $\beta\varphi$ subspace of l^1. For every sequence space S the $\beta\varphi$ space $Q = [\Lambda(T_n)S]$ verifies the following statements:*

1. *(T_nx) is bounded in Q for each $x \in S$;*

2. *$\Sigma\, s(n)T_nx$ converges to $(\Sigma\, s(n)T_n)x$ in Q for each $s \in \Lambda$ and $x \in S$;*

3. *Q has $AK(T_n)$ if and only if $B_x = \{T_kT_nx : k, n \in \mathcal{N}\}$ is bounded in Q for each $x \in S$;*

4. *Q has $AK(T_n)$ if $Q \subset \tilde{S}$.*

Note. As before, we obtain the cited portion of Garling's Theorem. If we take $(T_n) = (P_n)$, then $T_kT_n = T_{\min(k,n)}$ and by 1 and 3, each Q becomes an AK space. Now $\Lambda(T_n)$ is isomorphic to both Λ and the dense subspace $\mathfrak{R}$ of bv_0 consisting of the diagonals of members of $\Lambda(T_n)$, a $\beta\varphi$ subspace by Corollary 2.7 of [27] so that $R = \varphi + \mathfrak{R}$ is a $\beta\varphi$ AK subspace of bv_0, and clearly, each $Q = [\Lambda(T_n)S] = [RS]$. When $\Lambda = l^1$ we obtain $R = bv_0$ and the Garling-like result. In any case, we see that our specialization to $(T_n) = (P_n)$ is already covered in Theorem 5.1. However, one may easily construct a qd (T_n) which is not a dscs but is closed under matrix multiplication. For such, the present theorem and not its predecessor applies to show that each Q must be a $\beta\varphi$ $AK(T_n)$ space.

Proof. Since (T_n) is a qlt, each Q is well defined. We may assume Λ contains φ, since Q remains unchanged if Λ is replaced by the dense $\beta\varphi$ subspace $\Lambda + \varphi$ of l^1.

1. If $x \in S$, and A is a Q-bounded subset of φ, then $B = \{h_{x,u} : u \in A\}$ is $\sigma(\Lambda', \Lambda)$-bounded since $\Lambda(T_n)\, S \subset Q$. As a subset of c, the set B is bounded in sup norm by virtue of (**) in Section 4. That is, the scalar set $\{h_{x,u}(e_n) : u \in A,\ n = 1, 2, \ldots\} = \{\langle T_nx, u\rangle : u \in A,\ n = 1, 2, \ldots\}$ is bounded. Hence $(p_A(T_nx))$ is bounded by some $M > 0$.

2. Given $s \in \Lambda$ and $u \in A$, set $y = (\Sigma\, s(n)T_n)x$ and note that, by continuity, for each j,

$$\begin{aligned}\left|\left\langle y - \sum_{n=1}^{j} s(n)T_nx, u\right\rangle\right| &= |h_{x,u}(s - P_js)| = \left|\sum_{n>j} s(n)h_{x,u}(e_n)\right| \\ &\le \sum_{n>j} |s(n)|\,|\langle T_nx, u\rangle| \le \sum_{n>j} |s(n)|\,p_A(T_nx) \\ &\le M \sum_{n>j} |s(n)|.\end{aligned}$$

Thus $p_A(y - \Sigma_{n=1}^{j} s(n)T_nx) \le M\, \Sigma_{n>j} |s(n)|$, and the limit is 0, so that $(\Sigma_{n=1}^{j} s(n)T_nx)$ converges to y in Q.

3. If Q has $AK(T_n)$ then (T_n) is equicontinuous on Q and takes bounded sets onto bounded sets. Hence by 1, B_x is bounded in Q for each $x \in S$.

Conversely, suppose B_x is bounded in Q for each $x \in S$. Since (T_n) is a qut, $T_n e_j \to e_j$ in Q for each j, and Q has AD by 2. Thus Q has $AK(T_n)$ if $(T_n z)$ is bounded in Q for each $z \in Q$ (Theorems 3.2 and 3.3(a).) Given $x \in S$ and $s \in \Lambda$ with $\|s\|_1 = 1$, set $y = T_s x = (\sum s(n)T_n)x$. By 2 and Theorem 2.1, each

$$T_k y = \sum_n s(n) T_k T_n x$$

is in the closed absolutely convex hull of B_x. Thus contained in a bounded set, $(T_k y)$ is bounded in Q, and (T_k) is bounded at any point in the set $\varphi \cup \Lambda(T_n)S$ spanning Q. Therefore, (T_n) is bounded at all points $z \in Q$.

4. If $T \in \Lambda(T_n)$ then, since (T_n) is a qlt, T is row finite and maps S continuously into $\tilde{Q}$ by Theorem 2.1. Since $\tilde{Q}$ is a complete K space, the mapping has a continuous extension to $\tilde{S}$ also represented by the matrix T, i.e., $T\tilde{S} \subset \tilde{Q}$, so that $[\Lambda(T_n)\tilde{S}] \subset \tilde{Q}$. Now $Q \subset \tilde{S}$ implies $[\Lambda(T_n)Q] \subset [\Lambda(T_n)\tilde{S}]$, and we have $[\Lambda(T_n)Q] \subset \tilde{Q}$. Replacing S with Q and Q with $[\Lambda(T_n)Q]$ in 1 we conclude that $(T_n x)$ is bounded in $[\Lambda(T_n)Q]$ for each $x \in Q$. (Note: we could have substituted into 2 and used Theorem 2.3.) Clearly, $\tilde{Q}$ induces on any of its subspaces a topology weaker than the subspace's $\beta\varphi$ topology, so $(T_n x)$ is bounded in $\tilde{Q}$, and hence in Q, for each $x \in Q$. From 2, Q has AD, and $T_n e_j \to e_j$ in Q for each j because (T_n) is a qut. By Theorems 3.2 and 3.3, then, Q has $AK(T_n)$. ∎

EXAMPLE 5.5 Set $(T_n) = (P_n + S_n)$, where S_n has all zeros except for a square block of 1's with the upper left corner at $(n+1, n+1)$ and lower right corner at $(n+n^2, n+n^2)$. Obviously, (T_n) is a quasi-diagonal sectional convergence scheme. For each $x \in S = c_0$,

$$A = \{n^{-2} e_{n+1} : n \in \mathcal{N}\} \subset \varphi$$

is uniformly bounded on $(T_n x)$ by $\|x\|$, and thus by 2 of the theorem it follows that A is a Q-bounded subset of φ. Choose $x \in c_0$ such that

$$x(j) \geq n^{-1} \qquad \text{for } j \leq n + n^2$$

and observe that

$$|\langle T_n T_n x, n^{-2} e_{n+1} \rangle| \geq n^{-2}(n^2 n^2 n^{-1}) = n$$

for each n, so that the B_x of 3 is not bounded in Q, and Q does not have $AK(T_n)$.

We now have coordinatewise and matrix multiplier methods which allow us to start with a sequence space S and produce a $\beta\varphi$ $AK(T_n)$ space Q. What conditions on Q determine that S in its $\beta\varphi$ topology has $AK(T_n)$? In the case of matrix multipliers with (T_n) a qd, we will show that a $\beta\varphi$ AD space S has $AK(T_n)$ if and only if $Q = [\Lambda(T_n)S] \subset \tilde{S}$ for all (some) Λ. The corresponding statement for coordinatewise multiplication fails unless the multiplier space is chosen with care: The multiplier $R = \omega$ is a barrelled $AK(T_n)$ space with $Q = RS = \omega$ for both $AK(T_n)$ and non-$AK(T_n)$ choices of S; cf. Example 3.7.

We first see when S and Q are dense $\beta\varphi$ subspaces of $S + Q$.

THEOREM 5.6. *Suppose S is a $\beta\varphi$ space, (T_n) is a quasi-diagonal sectional convergence scheme, and Λ is a dense $\beta\varphi$ subspace of l^1. Set $Q = [\Lambda(T_n)S]$ and give $S + Q$ its $\beta\varphi$ topology.*

(a)(i) *S is a dense $\beta\varphi$ subspace of $S + Q$ if and only if* (ii) *(T_n) is equicontinuous on S.*

(b)(i) *Q is a dense $\beta\varphi$ subspace of $S + Q$ if and only if* (ii) *$S + Q$ has AD.*

Proof. (a)(i) $\Rightarrow$ (ii). Given $x \in S$, it follows by 1 of Theorem 5.4 that $(T_n x)$ is bounded in Q and, therefore, in the larger $S + Q$; and thus also in S, by (i). Theorem 3.2 applies.

(ii) $\Rightarrow$ (i). Suppose each $(T_n x)$ is bounded in S. Then each $\sum s(n)T_n x$ $(s \in \Lambda)$ is absolutely convergent to some $y \in \tilde{S}$ and to $z = (\sum s(n)T_n)x$ in Q by 2 of Theorem 5.4. Since both $\tilde{S}$ and Q are continuously included in ω, the series converges to both y and z in the Hausdorff space ω, so that $y = z$, and $S \subset S + Q \subset \tilde{S}$ implies (i).

(b)(i) $\Rightarrow$ (ii). By 2 of Theorem 5.4, Q is a dense $\beta\varphi$ subspace densely containing φ so that $S + Q$ has AD.

(ii) $\Rightarrow$ (i). Clearly $Q \supset \varphi$ is dense in the AD space $S + Q$. If A is a Q-bounded subset of φ and $x \in S$ then 1 of Theorem 5.4 yields a bound M for $(p_A(T_n x))$. Thus if $u \in A$ we have $|\langle T_n x, u\rangle| \leq M$ for each n. By Theorem 3.1, $(T_n x)$ converges to x in ω, and $|\langle x, u\rangle| = \lim_n |\langle T_n x, u\rangle| \leq M$, implying A is numerically bounded at x by M. Hence A is $S + Q$-bounded and Q is a $\beta\varphi$ subspace of $S + Q$. ∎

THEOREM 5.7. *Let S be a $\beta\varphi$ space, let (T_n) be a quasi-diagonal sectional convergence scheme, and let Λ be a dense $\beta\varphi$ subspace of l^1. The following statements are equivalent:*

1. *S has $AK(T_n)$;*
2. *S has AD and $\Lambda(T_n)S \subset \tilde{S}$;*
3. *$[\Lambda(T_n)S]$ is a dense $\beta\varphi$ subspace of $\tilde{S}$;*
4. *$[\Lambda(T_n)S]$ is a dense barrelled subspace of $\tilde{S}$, having $AK(T_n)$.*

Proof. $1 \Rightarrow 2$. Surely, S has AD and (T_n) is equicontinuous on S, so by Theorem 5.6(a), S is a dense $\beta\varphi$ subspace of $S + [\Lambda(T_n)S]$, ensuring $\Lambda(T_n)S \subset \tilde{S}$.

$2 \Rightarrow 3$. Since S has AD, so does $S + [\Lambda(T_n)S] \subset \tilde{S}$. By Theorem 5.6(b), $Q = [\Lambda(T_n)S]$ is a dense $\beta\varphi$ subspace of $S + Q \subset \tilde{S}$, and thus of $\tilde{S}$, via transitivity.

$3 \Rightarrow 1$ and $4 \Rightarrow 1$, easily: Theorem 5.4, part 4, guarantees $[\Lambda(T_n)S]$ has $AK(T_n)$ so that

$$\tilde{S} = [\Lambda(T_n)S]^{\sim}$$

has $AK(T_n)$ by Theorem 3.3, implying that S does as well.

To complete the proof we assume 3 holds and show $[\Lambda(T_n)S] = Q$ is barrelled. Indeed if G is a $\sigma(Q', Q)$-bounded subset of Q', then G is equicontinuous: Identify each $g \in G$ with the sequence $(g(e_n))$. Since Q has $AK(T_n)$, for $y \in Q$ we have $g(y) = \lim_n g(T_n y) = \lim_n \langle T_n y, g\rangle = \lim_n \langle y, T_n^{\top} g\rangle$. We need show only that the subset $C = \{T_n^{\top} g : g \in G, n = 1, 2, \ldots\}$ of φ is Q-bounded, for then we would have

$$|g(y)| = \lim_n \langle y, T_n^{\top} g\rangle \leq p_C(y)$$

for every $y \in Q$ and $g \in G$. Given $x \in S$ and $g \in G$, define $\kappa_{x,g}$ on Λ by

$$\kappa_{x,g}(s) = g\left(\left(\sum s(n)T_n\right)x\right) = \lim_k \left\langle\left(\sum s(n)T_n\right)x, T_k^{\top} g\right\rangle.$$

The linear functional $\kappa_{x,g}$ is continuous since it is the pointwise limit of linear functionals h_{x,u_k} that are equicontinuous by (**) where $u_k = T_k^{\top} g$. Because G is bounded at any $(\sum s(n)T_n)x \in Q$, the set $\{\kappa_{x,g} : g \in G\}$ is bounded at any $s \in \Lambda$. Let each $g \in G$ also denote its continuous extension to $\tilde{Q} = \tilde{S}$. Since S has $AK(T_n)$, $\lim_n \kappa_{x,g}(e_n) = \lim_n g(T_n x) = g(x)$, and $\{\kappa_{x,g} : g \in G\}$ may be viewed as a subset of c that, by (**), is bounded in the sup norm. That is, for fixed $x \in S$ the set of scalars of the form $\kappa_{x,g}(e_n) = g(T_n x) = \langle x, T_n^{\top} g\rangle$ is bounded. Therefore, $C \subset \varphi$ is bounded at each $x \in S$, hence at each member of $\tilde{S} = \tilde{Q}$. ∎

Notes. a. If S is assumed to be barrelled instead of $\beta\varphi$, the result still holds in light of Corollary 3.5, provided we insert "S has its $\beta\varphi$ topology and" at the beginning of each of 2–4, so that $\tilde{S}$ is defined.

b. If (T_n) is an arbitrary gscs and (1) $\Lambda(T_n)S$ is still defined, (2) each $(T_n e_i)_n$ still converges to e_i in S, and (3) we still have $\varphi \subset S^{(\beta)}$ then, parts 1 and 2 of Theorem 5.4 still hold (same proof) and quickly show the equivalence of parts 1 and 2 of Theorem 5.7 in this setting via Theorem 3.3.

c. If S is a $\beta\varphi$ $AK(U_n)$ space for some gscs (U_n), then there exists a gscs (T_n) such that S has $AK(T_n)$ but $\Lambda(T_n)S$ is not defined for some (all) Λ if and only if S is non-AK. Moreover, we can choose (T_n) to be a quasi-upper triangular gscs (qut) if (U_n) is. The proof is a separate effort. Thus the non-AK space S in Example 5.2 has $AK(V_n)$ for some qut (V_n) such that $\Lambda(V_n)S$ is undefined for all dense barrelled subspaces Λ of l^1. Obviously, this permits us to rewrite Theorem 5.7 with S an AK space and (T_n) an arbitrary qut: *Let S be a dense barrelled subspace of l^1. S has $AK(T_n)$ if and only if $\Lambda(T_n)S$ is a well-defined subset of S.*

The hypothesis that S is a $\beta\varphi$ space corresponds to the hypothesis of Kadec and Pełczynski that the biorthogonal system (e_n, E_n) be norming. Under this hypothesis they proved, in our terminology, that for a $\beta\varphi$ AD FK space S and $s \in S$ the following assertions are equivalent: (a) $m_0 s \subset S$ where m_0 is the space of finitely valued sequences; (b) $ms \subset S$; (c) $\sum_n E_n(s)e_n$ converges unconditionally to s (Theorems 4 and 5 of [14].) By using a version of the Closed Graph Theorem, Bachelis and Rosenthal [1] eliminated the hypothesis that the system be norming, i.e., that the FK space S be a $\beta\varphi$ space. In a setting more general than FK spaces, we will remove the $\beta\varphi$ hypothesis from Theorem 5.7. However, Corollary 3.5 shows that under the conditions of our new Theorem 5.8, the space S will still have its $\beta\varphi$ topology.

THEOREM 5.8. *Let S be a barrelled metrizable K space, and let M be a dense barrelled subspace of l^1 (T_n) for some quasi-diagonal sectional convergence scheme (qd) (T_n). The following statements are equivalent:* (A) *S has $AK(T_n)$*; (B) *S has AD, has a K space completion $\hat{S}$, and $MS \subset \hat{S}$*; (C) *S has a K space completion $\hat{S}$, and $[MS]$ is a dense barrelled subspace of $\hat{S}$ having $AK(T_n)$.*

Proof. The normed spaces M and $\Lambda(T_n)$ are identical, where

$$\Lambda = \left\{ s \in l^1 : \sum s(n)T_n \in M \right\}$$

is a dense subspace of l^1 that is barrelled by Proposition 4.1. Denote by T the space S endowed with its coarser $\beta\varphi$ topology (i.e., the $\beta(\cdot, \varphi)$ topology). If S has a K space completion $\hat{S}$, it is continuously included in $\tilde{T}$.

(A) $\Rightarrow$ (B). This follows from ($1 \Rightarrow 2$) of Theorem 5.7 since (A) and Corollary 3.5 imply $T = S$.

(B) $\Rightarrow$ (C). If S has AD, so does T, and $MT \subset \hat{S} \subset \tilde{T}$ ensures, by ($2 \Rightarrow 4$) of Theorem 5.7, that $[MT]$ is a dense barrelled subspace of $\tilde{T}$ having $AK(T_n)$. Thus $[MT]$ is a dense barrelled subspace of U, where U

denotes $\hat{S}$ with its coarser $\beta\varphi$ topology, and U is barrelled. By the Closed Graph/Open Mapping Theorem U coincides with the Fréchet space $\hat{S}$, and (C) follows.

(C) $\Rightarrow$ (A). Since a dense barrelled subspace of $\hat{S}$ has $AK(T_n)$, so does $\hat{S}$, by Theorem 3.3, and hence so must S. ∎

Remarks. We cannot omit either part of the hypothesis that (T_n) be a qut and qlt; i.e., a qd. By Note c there exists an *AD BK* space S and a qut (T_n) such that (A) holds and (B) fails. Example 3.10 provides an *AD BK* space S and a qlt (T_n) such that (B) holds and (A) fails.

In our experience with Theorems 5.7 and 5.8 we find that knowing the result (Theorem 5.7) under the assumption that S has its $\beta\varphi$ topology allows us to quickly achieve the result (Theorem 5.8) when S is not assumed to have its $\beta\varphi$ topology but, instead, is taken to be metrizable and barrelled. (In the literature, S is typically taken to be an *FK* space.) And since, in light of Corollary 3.5., we cannot avoid the $\beta\varphi$ topology on an *FK* space if it is to have $AK(T_n)$, our strategy in Theorem 5.7 reverses that of Bachelis and Rosenthal [1]. Rather than seeking to submerge the irrepressible $\beta\varphi$ hypothesis we made it the *only* requirement on S. By contrast, the classical *FK* hypothesis seems quite restrictive.

References

1. G. F. Bachelis and H. P. Rosenthal, On unconditionally converging series and biorthogonal systems in Banach spaces, *Pacific J. Math.* **37** (1971), 1–5.
2. G. Bennett, Some inclusion theorems for sequence spaces, *Pacific J. Math.* **46** (1973), 17–30.
3. G. Bennett and N. J. Kalton, Inclusion theorems for *K*-spaces, *Canad. J. Math.* **25** (1973), 511–523.
4. G. Bennett and N. J. Kalton, *FK* spaces containing c_0, *Duke Math. J.* **39** (1972), 561–582.
5. G. Bennett and N. J. Kalton, Addendum to *FK* spaces containing c_0, *Duke Math. J.* **39** (1972), 819–821.
6. M. Buntinas, On Toeplitz sections in sequence spaces, *Math. Proc. Cambridge Philos. Soc.* **121** (1975), 451–460.
7. M. Buntinas and G. Goes, Products of sequence spaces and multipliers, *Rad. Mat.* **3** (1987), 287–300.
8. J. Dieudonné, On biorthogonal systems, *Michigan Math. J.* **2** (1954), 7–20.
9. P. Enflo, A counterexample to the approximation property in Banach spaces, *Acta Math.* **130** (1973), 309–317.
10. D. J. H. Garling, On topological sequence spaces, *Proc. Cambridge Philos. Soc.* **63** (1967), 997–1019.
11. D. J. H. Garling, The β- and γ-duality of sequence spaces, *Proc. Cambridge Philos. Soc.* **63** (1967), 963–981.
12. V. F. Gaposhkin and M. I. Kadec, Operational bases in Banach space, *Math. Sb.* **61** (103), 3–12. [Russian]

13. M. I. KADEC, Bases and their spaces of coefficients, *Dopovidi Akad. Nauk Ukrain. RSR Ser. A* **9** (1964) 1139–1141. [Ukrainian]
14. M. I. KADEC AND A. PEŁCZYNSKI, Basic sequences, biorthogonal sequences and norming sets in Banach and Frèchet spaces, *Studia Math.* **25** (1965), 297–323. [Russian]
15. G. KÖTHE AND O. TOEPLITZ, Lineare Räume mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen, *J. Reine Angew. Math.* **171** (1934), 193–226.
16. C. W. MCARTHUR, Developments in Schauder basis theory, *Bull. Amer. Math. Soc.* (*N.S.*) **78** (1972), 877–908.
17. R. J. MCGIVNEY AND W. RUCKLE, Multiplier algebras of biorthogonal systems, *Pacific J. Math.* **29** (1969), 375–387.
18. G. MEYERS, On Toeplitz sections in *FK* spaces, *Studia Math.* **51** (1974), 23–33.
19. W. ROELCKE AND S. DIEROLF, On the three-space problem for topological vector spaces, *Collect. Math.* **32** (1981), 3–25.
20. W. RUCKLE, On the characterization of sequence spaces associated with a Schauder basis, *Studia Math.* **28** (1967), 279–288.
21. W. RUCKLE, Lattices of sequence spaces, *Duke Math. J.* **35** (1968), 491–503.
22. W. RUCKLE, Representation and series summability of complete biorthogonal sequences, *Pacific J. Math.* **34** (1970), 511–528.
23. W. RUCKLE, Topologies on sequence spaces, *Pacific J. Math.* **42** (1972), 491–504.
24. W. RUCKLE, The strong φ topology on symmetric sequence spaces, *Canad. J. Math.* **37** (1985), 1112–1133.
25. W. RUCKLE, FK spaces in which the sequence of coordinate vectors is bounded, *Canad. J. Math.* (1973), 973–978.
26. S. SAXON, Some normed barrelled spaces which are not Baire, *Math. Ann.* **209** (1974), 153–160.
27. S. SAXON, Non-barrelled dense $\beta\varphi$ subspaces, *J. Math. Anal. and Appl.*, to appear.
28. S. SAXON AND M. LEVIN, Every countable codimensional subspace of a barrelled space is barrelled, *Proc. Amer. Math. Soc.* **29** (1971), 91–96.
29. S. SAXON AND P. P. NARAYANASWAMI, (*LF*)-spaces, quasi-Baire spaces and the strongest locally convex topology, *Math. Ann.* **274** (1986), 627–641.
30. H. H. SCHAEFER, "Topological Vector Spaces," Macmillan Co., New York, 1966.
31. G. L. SEEVER, Measures on *F*-spaces, *Trans. Amer. Math. Soc.* **133** (1968), 267–280.
32. I. SINGER, "Bases in Banach Spaces I, II," Springer-Verlag, Berlin/New York, 1970.
33. A. WILANSKY, "Modern Methods in Topological Vector Spaces," McGraw–Hill, New York, 1978.
34. S. YAMAZAKI, Normal rings and bases in Banach spaces, *Sci. Papers College Gen. Ed. Univ. Tokyo* **16** (1966), 25–26.
35. K. ZELLER, Abschnittskonvergenz in *FK*-Räumen, *Math. Z.* **55** (1951), 55–70.